Capital allocation beyond Euler

108. Mitgliederversammlung der SAV

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- Capital allocation for portfolios
- Capital allocation on risk factors
- Case study

Why capital allocation?

- "Just" calculating solvency capital is not enough!
  - Capital requirement needs to be understood and integrated into business and strategy.

- Capital allocation splits the total required/target capital C into amounts $C_1, \dots, C_n$ with

$$C = \sum_{i=1}^{n} C_i$$

where each C_i is an amount of capital related to a risk factor or part of the business.

- Capital allocation is a tool to answer important questions about your business:
  - What are your greatest risks?
  - What are the sources of diversification?
  - Are you adequately rewarded for the risks you take?
  - How can you optimise risk-return?

- Under Solvency II it is required as part of the use test and the ORSA

Capital allocation for portfolios of risk

- The capital allocation for a portfolio of risks is the most important special case of allocation.
- Portfolio of risks means the total P&L or loss function is a sum:

$$\mathrm{TOT} = \sum_{i=1}^{n} X_i$$

TOT: Total P&L or total loss

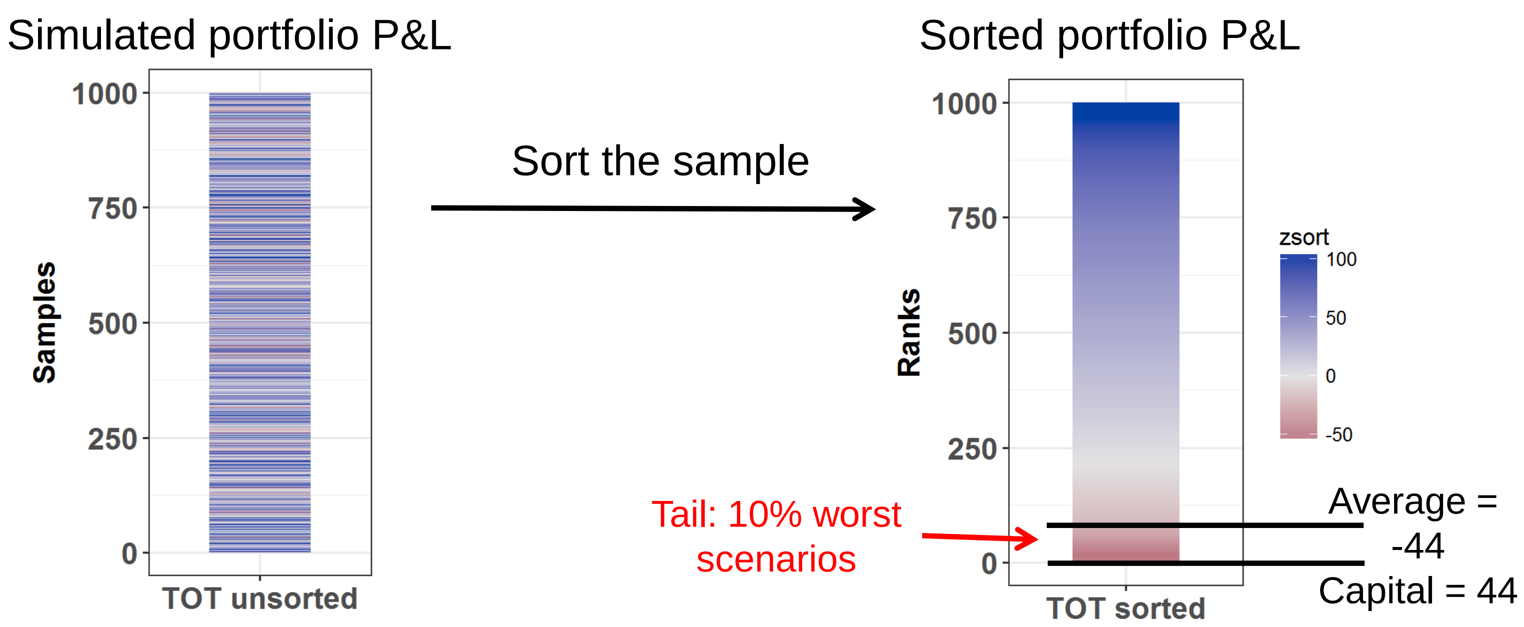
X_i: P&L or Loss of portfolio components, risk factors

- Euler method: Method to allocate capital C_i to the components X_i of a portfolio of risks
  - Has very nice properties
  - Easy to calculate (for many risk measures)
  - Intuitive interpretation (for many risk measures)

- There are many examples of portfolio of risks where the Euler method is used in practice
  - Allocation to financial instruments in an investment portfolio
  - Allocation to insurance contracts in an insurance portfolio
  - Allocation to lines of business
  - Allocation to legal entities of a group

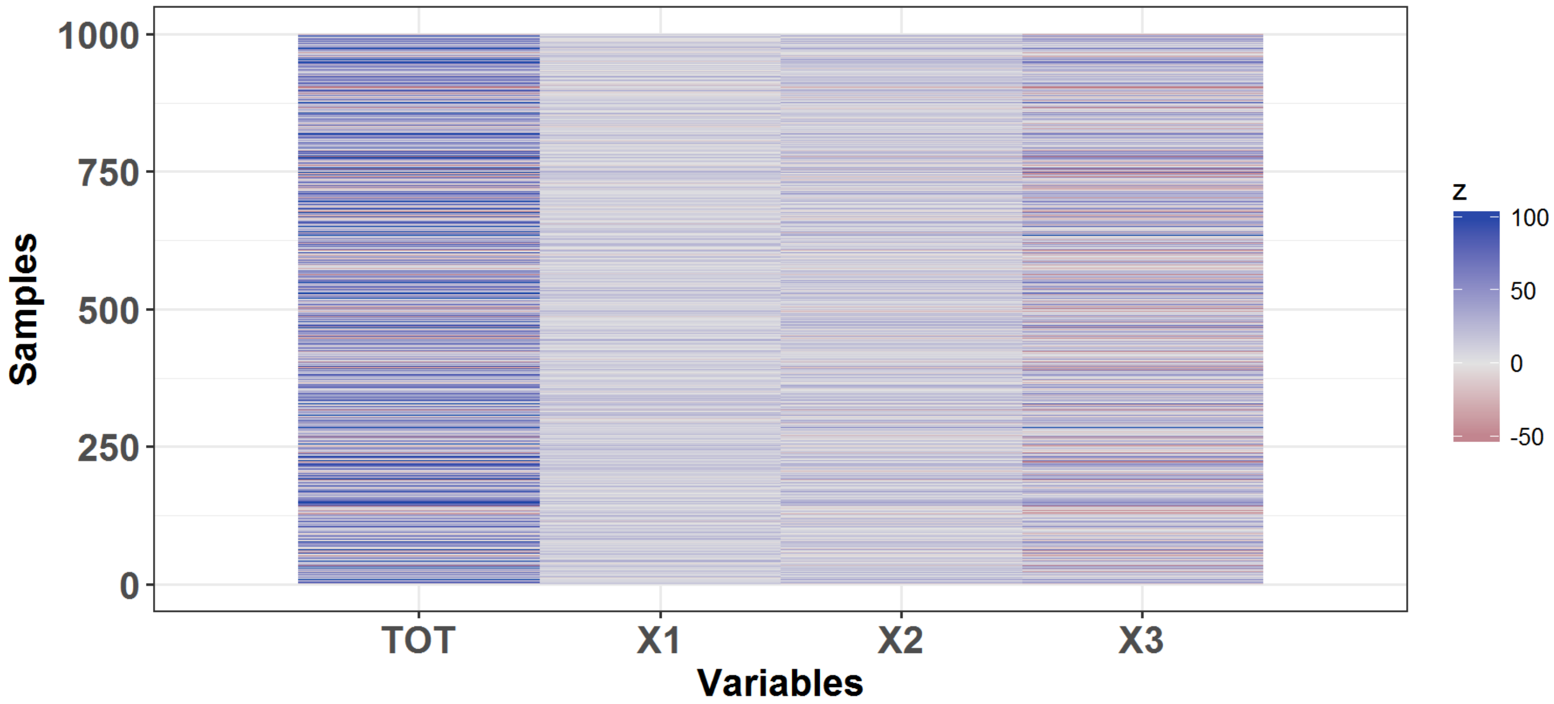
Example: Expected Shortfall

- The risk measure Expected Shortfall allows a particularly nice Euler allocation.
- Expected Shortfall is estimated as average of worst outcomes of a simulation. In the figure at 10% level: $C = -\mathrm{E}[TOT \mid TOT < q_{10\%}]$



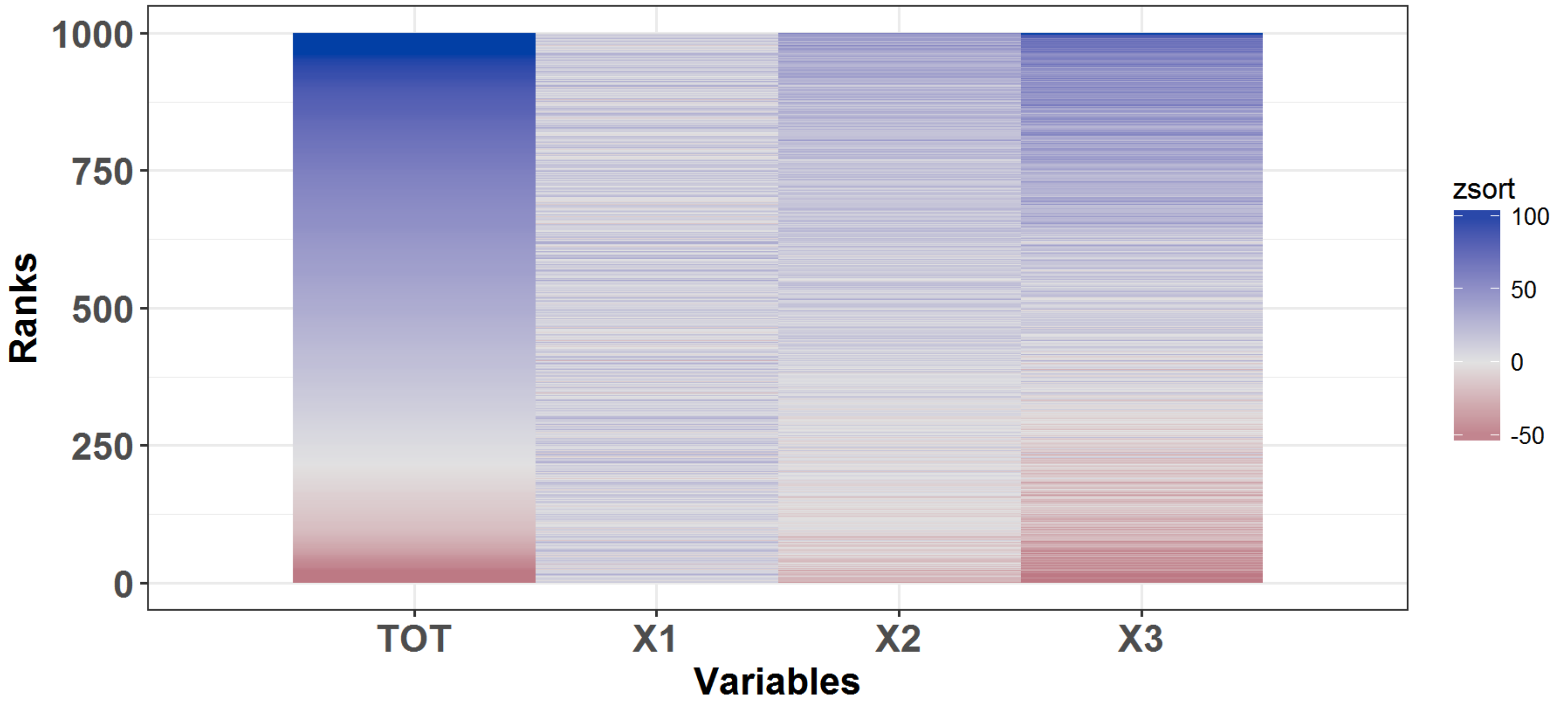
Example: Joint simulation

- Example: A portfolio of three risks with $\mathrm{TOT} = X_1 + X_2 + X_3$
  - Joint simulation with $N = 1000$ of the P&L of the four variables.
  - Each row is an independent sample.
  - Each column a variable.



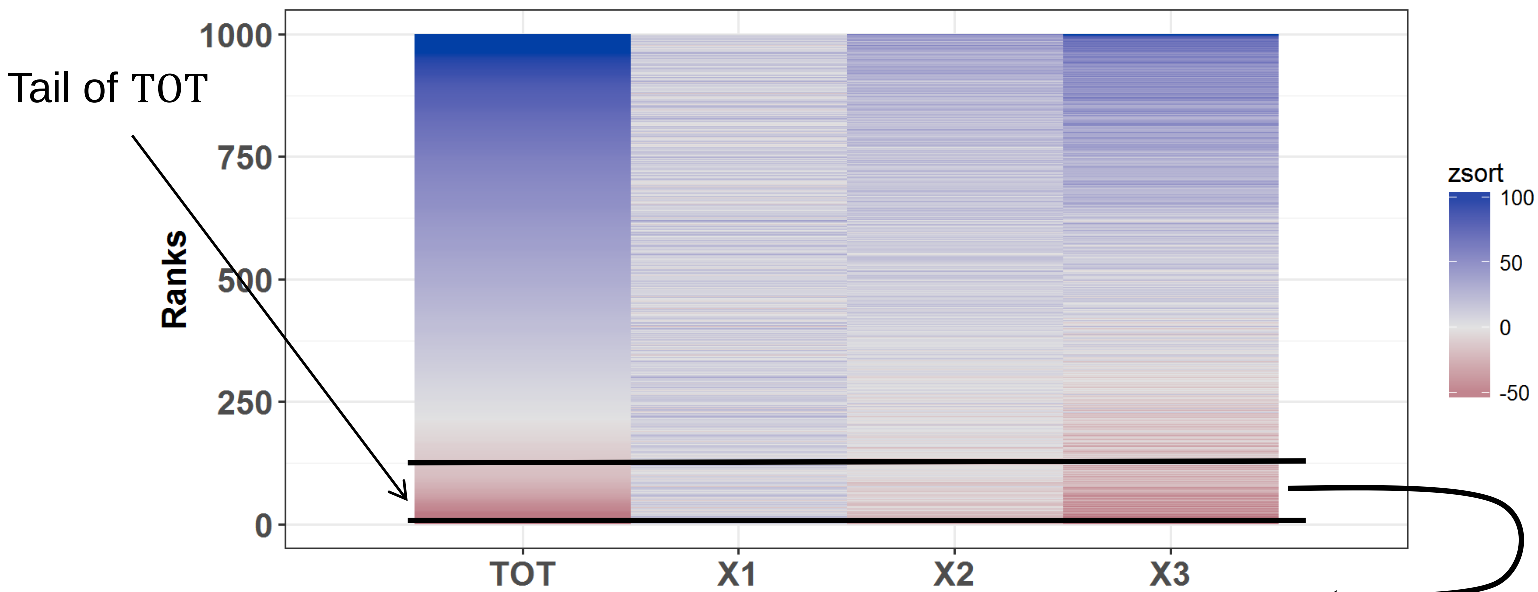
Example: Sorted outcomes

- Sort rows according to TOT the total P&L: Good outcomes of TOT on top bad ones at the bottom.
  - X3 and (to a lesser extent) also X2 are bad if TOT is bad.
  - X1 seems to be undetermined.



Example: Allocation of Expected Shortfall

The Euler allocation for X1, X2 and X3 is their tail average according to the sort order of TOT.
Total capital: $C = 44$ allocated capital: $C_1 = -8$ $C_2 = 12$ $C_3 = 40$



-44 = 8 + -12 + -40 Average values

$$C = C_1 + C_2 + C_3$$

$$\mathrm{E}[TOT \mid TOT < q_{10\%}] = \mathrm{E}[X_1 \mid TOT < q_{10\%}] + \mathrm{E}[X_2 \mid TOT < q_{10\%}] + \mathrm{E}[X_3 \mid TOT < q_{10\%}]$$

Euler allocation always sums up to total capital!

Euler allocation as a useful tool

- The Euler allocations has nice properties:
  - Allocated capital sums up to total capital
  - Allocation can be computed from simulations
  - Intuitive interpretation

- Euler is the only method which provides all the answers:

  - Largest risk? $\Rightarrow$ Risk factor with largest allocated capital
  - Diversification? $\Rightarrow$ Allocated capital smaller than stand-alone capital
  - Measure reward? $\Rightarrow$ Return On Risk Adjusted Capital (RORAC) Expected return (total or component) divided by (total or allocated) capital.
  - Optimisation? $\Rightarrow$ RORAC compatibility: Increasing exposure to component with largest component-RORAC will increase RORAC of total portfolio

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BUT: Not all risks come as a portfolio!

- Portfolios of risks are common but there are many examples where risk factors combine in a non-linear fashion.
- Discounted or FX cash flows $f(X,Y) = X \cdot Y$

  X (insurance) cash flow

  Y discount factor or FX rate
- Excess of loss treaty with multiple perils

  $f(X,Y) = \max(X + Y - c, 0)$

  X, Y perils e.g. earthquake, hurricane
  c deductible
- Example: Financial return guarantee on a mixed investment portfolio

  $f(X,Y) = \max(X + Y - c, 0)$

  X, Y asset classes, c guarantee/strike level
- How does capital allocation actually work in those cases?
- In these cases there is currently no “gold-standard” for allocation comparable to Euler allocation.

What is the problem?

- Immediately obvious algebraic problem:
  $\mathrm{E}[TOT \mid TOT < q_{10\%}] = \mathrm{E}[X_1 \mid TOT < q_{10\%}] + \mathrm{E}[X_2 \mid TOT < q_{10\%}] + \mathrm{E}[X_3 \mid TOT < q_{10\%}]$
  works only for $TOT = X_1 + X_2 + X_3$.

- Deeper conceptual problem:
  - The marginal principle $C[X_i] = C[TOT] - C[TOT - X_i]$ breaks down because $TOT - X_i$ has no meaning for non-additive risk factors.
  - Euler principle is infinitesimal version of the marginal principle

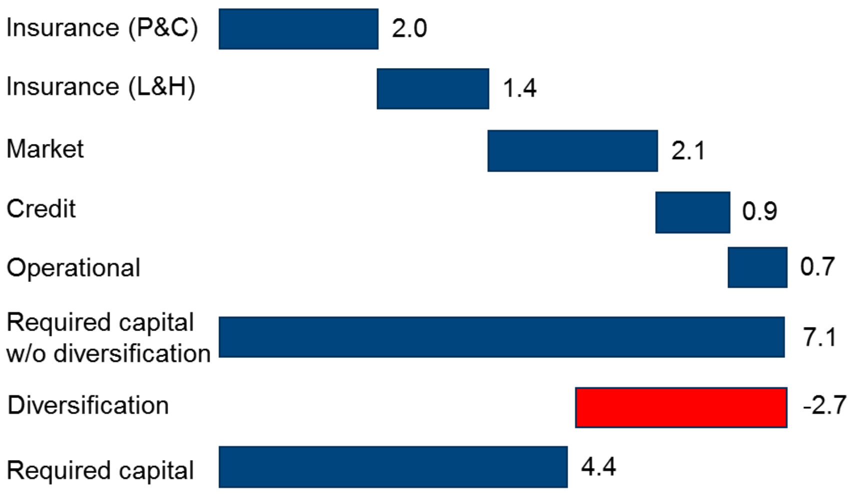
- From a business perspective:
  - Euler allocation is closely related to what you can actually DO with a portfolio: Increase/Decrease the exposures to the single risk factors.
  - When discounting a cash-flow you can't increase/decrease the exposure to the discount factor.
  - If you can't change the exposure RORAC compatibility is pretty useless

What can be done?

- Loss allocation according to the Cat model vendors: Allocate loss in a simulation year to the risk factor (event) which causes the bond/insurance contract to trigger.
  - Works only for event type risk factors
  - Ignores interaction of events (for example: Aggregate covers)
  - Has poor statistical qualities

- Split by risk category
  - Capital per risk category is routinely reported.
  - But risk factors such as interest (or FX) rates enter into all lines of business and investments. How are they carved out from the rest?
  - What does “diversification” mean?
  - Can this serve as a basis for capital allocation?

Generic example of a split by risk category



Split by “Freezing-the-Margins”

- Split by freezing the margins might be the most popular method to calculate capital per risk factor. Example: Split capital for a P&L model $f(X,Y)$ with risk factors insurance risk (X) and market risk (Y) into capital for insurance and market risk.
- Step 1: Define “pure insurance risk” by replacing all stochastic inputs Y for market risk with a constant value y_0: $INS(X) = f(X, y_0)$
- Step 2: Define “pure market risk” by replacing X with the constant value x_0:
$$MKT(Y) = f(x_0, Y)$$
- Step 3: Run the model three times to calculate the “stand-alone” capitals for INS and MKT and the total risk TOT.
  - Capital for insurance risk $C_{INS} = C[INS(X)] = C[f(X, y_0)]$
  - Capital for market risk $C_{MKT} = C[MKT(Y)] = C[f(x_0, Y)]$
  - Total capital $C = C_{TOT} = C[f(X, Y)]$
- Step 4: Add up and call the difference “diversification”

$$C_{TOT} = C_{INS} + C_{MKT} - \text{Diversification}$$

Split by freezing-the-margins seems to be quite intuitive but has three problems!

The problems with freezing-the-margins

- First problem: The “pure” models do not add up!

$$f(X, Y) \neq f(x_0, Y) + f(X, y_0)$$

- Solution: A residual term needs to be included in the allocation

$$f(X, Y) = f(x_0, Y) + f(X, y_0) + RES$$ Split of C_{TOT} into $C_{INS}, C_{MKT}, C_{RES}$

- Second problem: The allocated capitals do not add up to the total capital.
- Solution: Use Euler allocation instead of stand-alone capital.

- Third problem: What do the terms $INS(X) = f(X, y_0)$ and $MKT(Y) = f(x_0, Y)$ represent in terms of business or in terms of modelling?
  - The terms have no consistent interpretation in terms of business
  - Lack of interpretation makes the choice of constants x_0, y_0 and the capital split arbitrary.
  - Simply replacing a random variable with a constant is not a consistent stochastic approach

A general framework

- Step 1: Split the total into a sum of components each depending on one single risk factor only – the "pure risk" functions – and the residual .

$$f(X,Y) = INS(X) + MKT(Y) + RES(X,Y)$$

- Step 2: Use Euler allocation to allocate capital onto each component.

$$f(X,Y) = INS(X) + MKT(Y) + RES(X,Y)$$

Euler allocation ↓ ↓ ↓

$$C = C_{INS} + C_{MKT} + C_{RES}$$

- The hard problem is the split into a sum, i.e. Step 1!
- The split should be based on principles
  - Principle 1: A split should be based on real world business considerations
  - Principle 2: A split should be mathematically sound and consistent

Split by optimal hedging

- The mathematical idea of split by optimal hedging is: Approximation.
  - Choose the pure models such that the residual term RES is as small as possible:

Find h and g such that
$$f(X,Y) - h(X) - g(Y) \rightarrow \text{minimal}$$

- The business idea behind split by optimal hedging is …. optimal hedging (or optimal reinsurance).
  - $MKT(Y)$, the optimal $g(Y)$, is the best hedge of the total P&L $f(X,Y)$ using only market risk instruments .
  - $INS(X)$, the optimal $h(X)$, is the best reinsurance of the total P&L $f(X,Y)$ using only reinsurance contracts not mentioning market risk.
  - $RES(X,Y)$ is the remaining basis risk.

Concrete implementation: Variance hedging

- Some specifications are required to turn split by hedging into a practical approach
  - What is the universe of permitted hedges or reinsurance contracts?
  - What is the metric to determine “optimal”?
  - How can these be calculated in practice?

- Metric: minimal variance (least squares)
  - Optimal solutions are conditional expectations, i.e. the mathematics is sound and well understood.

- Permitted instruments/pure models
  - Choice depends on f and practical considerations
  - Typically parametric families (see next section)

- Practical calculations
  - Least squares is easy using regression techniques
  - Big advantage: Just a single model run required no matter how many risk factors there are in the split.

Does the method make a difference?

- It is not difficult to test typical functions over a range of relevant distributional assumptions and compare the results of the various splitting methods.
- Some observations for $f(X,Y) = X \cdot Y$
  - The residual term in the split freeze can be substantial (>20% of total capital) especially for correlated risk factors
  - For independent risks split freeze and variance hedging are exactly identical
  - For correlated risks they are different, differences can be 10% of total capital or more
  - One of the causes of differences is cross-hedging of correlated risk, which is ignored by the freeze approach
- Some observations for $f(X,Y) = \max(X + Y - c, 0)$
  - Behaviour for the freeze method depends strongly on interplay between deductible c and the frozen points x_0, y_0.
  - For low deductibles f is like $X + Y$ and freeze and variance methods produce similar results.
  - For higher deductibles residual terms can get very large
  - Freeze for higher deductibles seems quite erratic (allocating 0% or 100%)
  - Differences between methods for high deductibles are huge

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The cat bond index

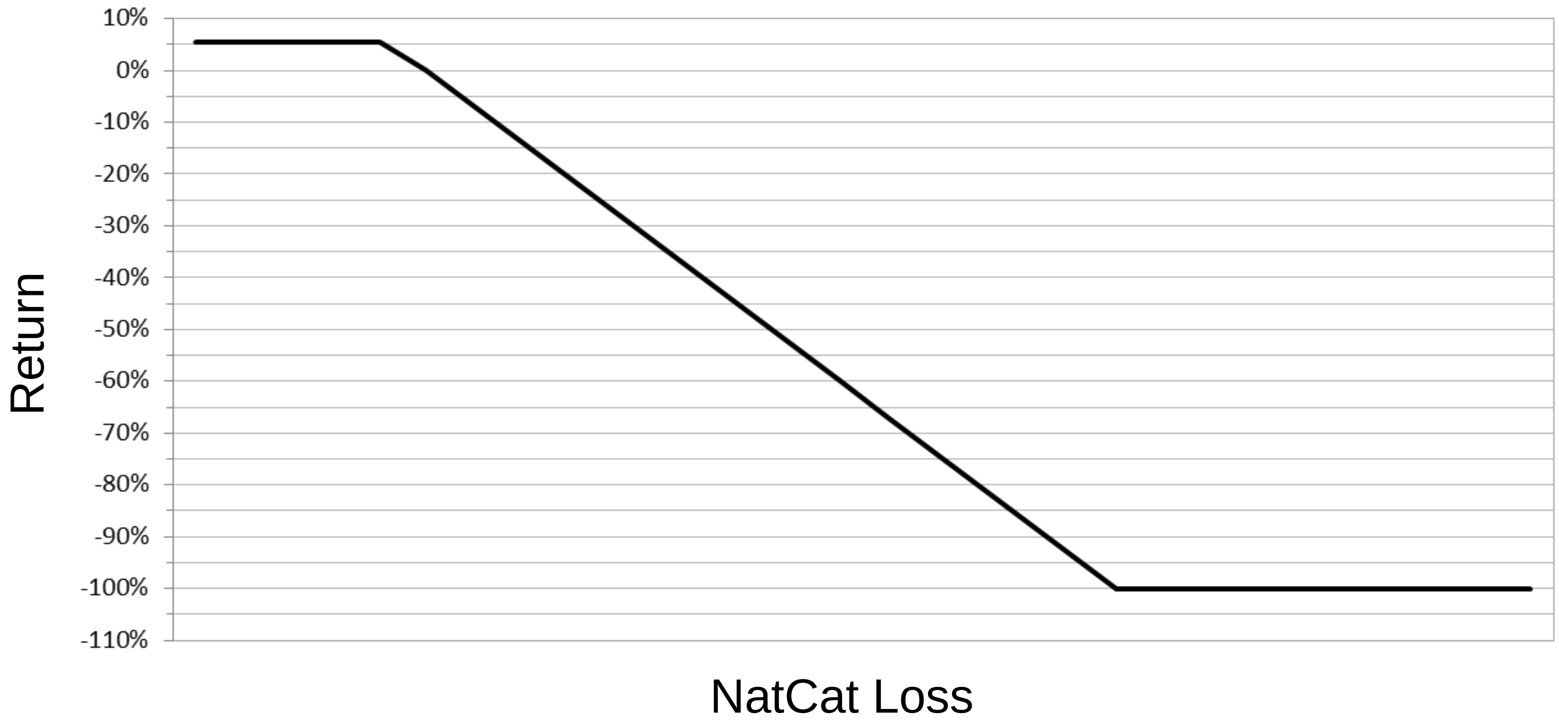
- **This case study is joint work with Jiven Gill from Schroders investment!**

- Swiss Re Global Cat bond index:
  - A portfolio of cat bonds designed to reflect the returns of the catastrophe bond market
  - Swiss Re Capital Markets launched the Index in 2007
  - First total return index for the sector.

- The question: "What are the largest risks contributing to losses for the Swiss Re Cat Bond index?"

Cat bond pay-out is non-linear

- Pay-out profile of a Cat bond on some kind of loss from natural catastrophes



The challenge: Cat bonds are not “pure risk”

- Cat Bond payoffs can depend on more than one type of natural disaster (peril)
  - Return $f(x,y,z)$ might depend on x: California earthquake losses, y: Florida Hurricane losses, z : European windstorm losses
  - Depending on the functional form $f(.)$, cat bond can be triggered due to losses from only one of the perils or from a combination of them.
  - Over 40% of the cat bonds in the Swiss Re Index are multi-peril bonds.

- The answer in four steps:
  - Step 1: Find “pure risk” functions to describe cat bonds returns
  - Step 2: Split each individual cat bond into a sum of “pure risk” functions
  - Step 3: Define the cat bond index as the weighted sum of the individual cat bonds “pure risk” functions
  - Step 4: Use Euler allocation of Expected Shortfall

Definition of the pure risk functions

- Parametric families of simple single peril instruments ("calls") are the building blocks of the pure risk functions:

$g_i(X) = \max(X - c_i, 0)$ X: denotes industry losses due a single peril such as industry loss from Florida Tropical Cyclone
c_i: deductible or attachment level of instrument i

- The pure risk functions are constructed from linear combinations fitted by ordinary least squares

$$d_X(X) = \sum_{i=1}^{n} \beta_i * \max(X - c_i, 0)$$

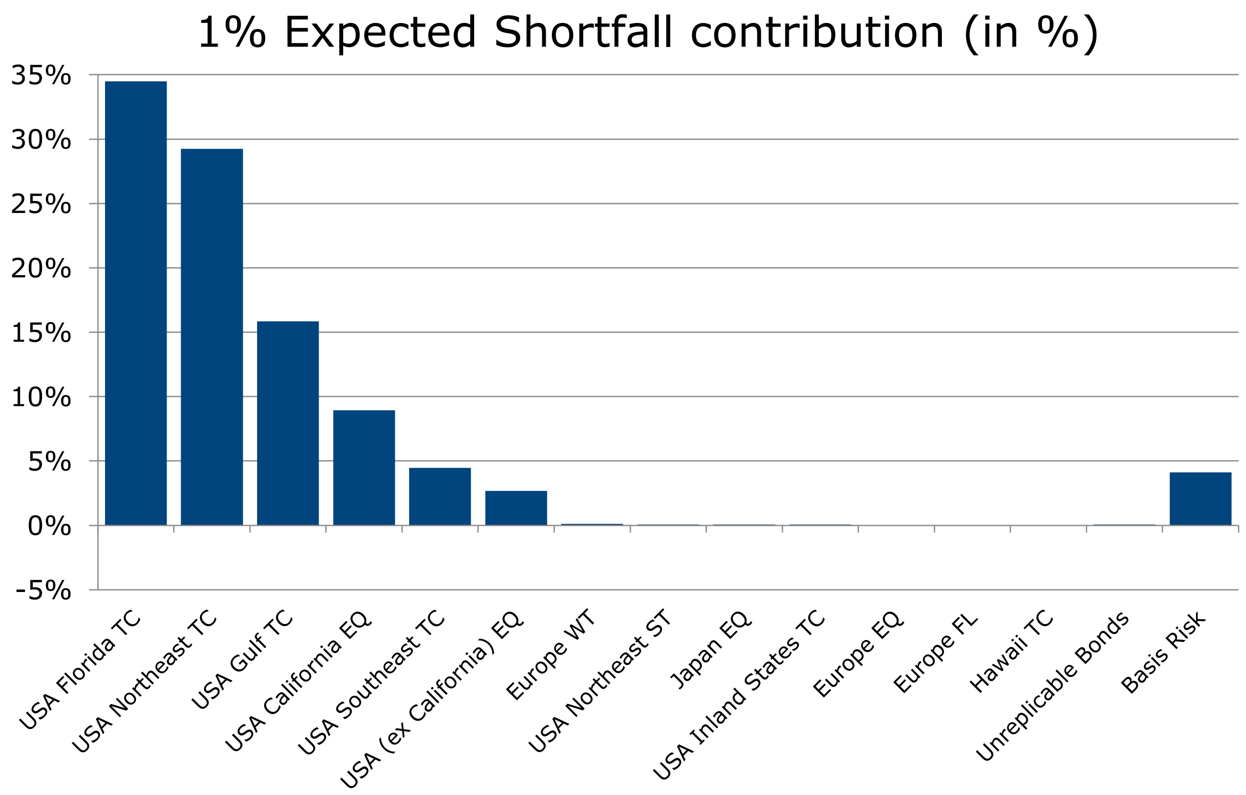
- There are pure risk functions for all perils/regions to replicate all bonds

$$f(X, Y, Z, \dots) = d_X(X) + d_Y(Y) + d_Z(Z) + \cdots + RES(X, Y, Z, \dots)$$

- Industry losses per perils and regions for calibration were extracted from AIR Catrader®

Allocation of Expected Shortfall

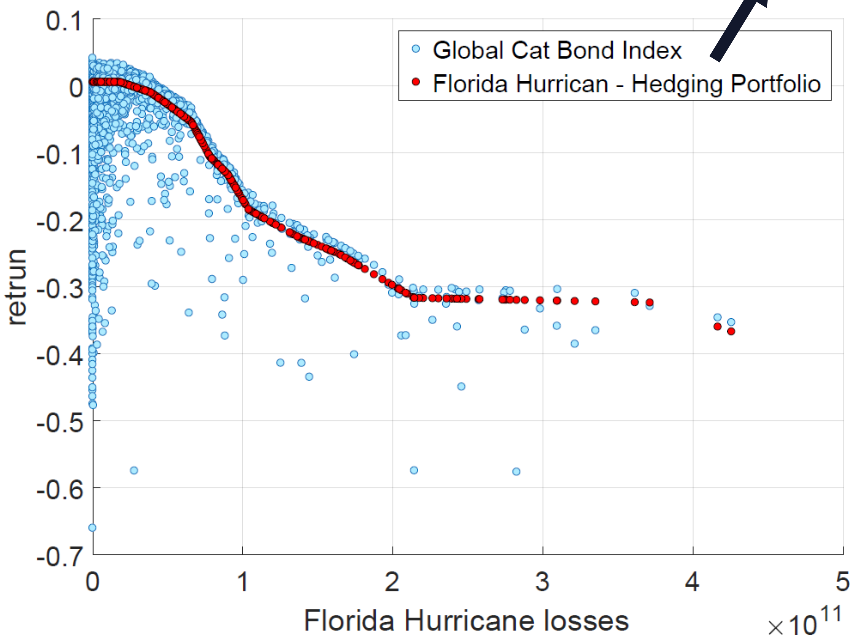
- A model of “pure” risk functions which adds up to 100%
- Each individual risk factor in the model has a business and economical meaning.

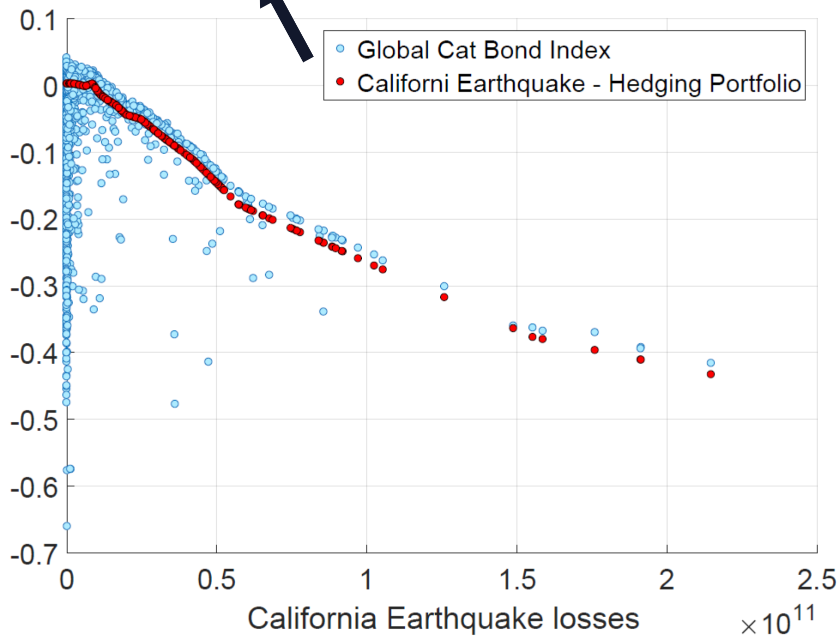


Cat Bond index as sum of pure risk functions

- The decomposition allows analysis beyond loss allocation

$$R_{Cat\ Bond\ index} = R_{Florida\_TC} + R_{California\_EQ} + \ldots + RES$$

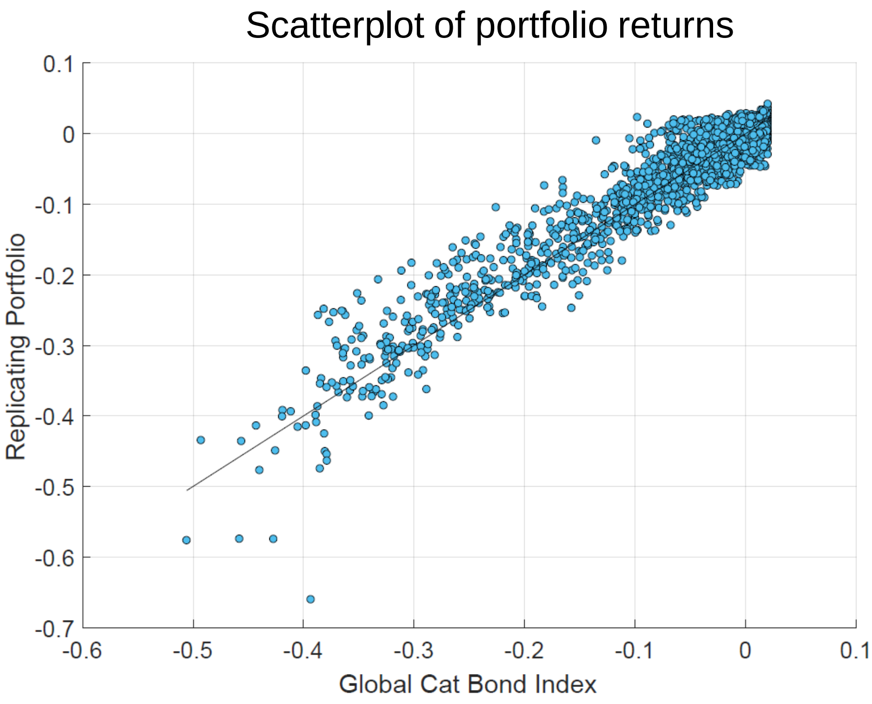


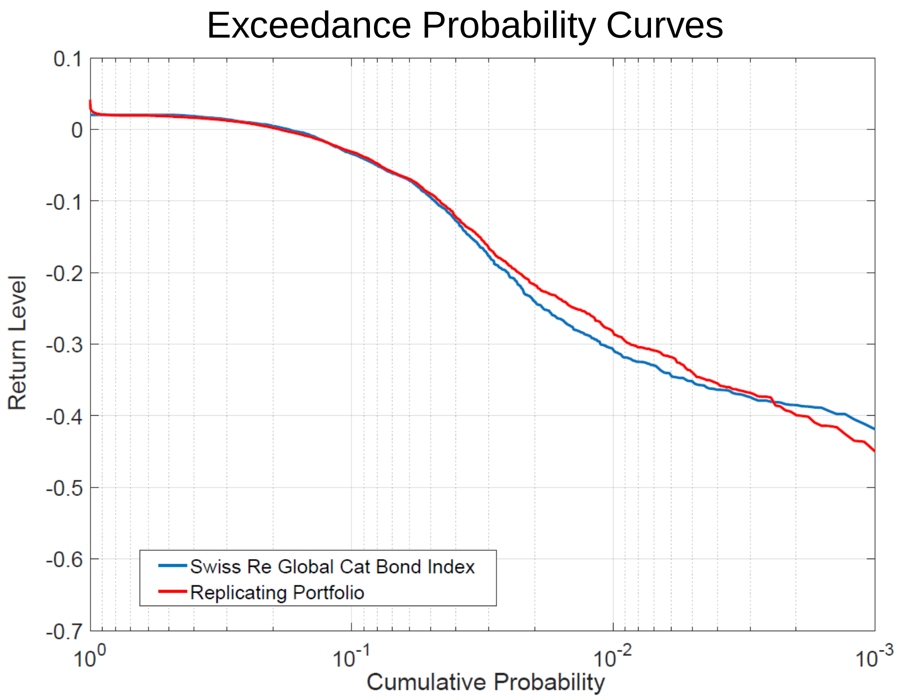


Red points are the pure risk functions

The Cat Bond index decomposed

- Overall fit is reasonably well even though there are two sources of error:
  - Errors due to the payoff function: $f(x,y) \neq f_1(x) + f_2(y)$
  - Errors due to risk factors: The pure risk instruments are based on *industry losses*, while bonds might insure company specific portfolios or have parametric triggers.





Further reading

- Find below some papers on the topic. But be warned: The literature is (still) quite technical!

- ”Decomposing life insurance liabilities into risk factors” (2015)
  Schilling, K., Bauer, D., Christiansen, M., Kling, A.,
  https://www.uni-ulm.de/fileadmin/website_uni_ulm/mawi2/dokumente/preprint-server/2016/2016_-_03.pdf
- “Risk Capital Allocation and Risk Quantification in Insurance Companies”(2012)
  Ugur Karabey, http://hdl.handle.net/10399/2566
- “Risk factor contributions in portfolio credit risk models”(2010)
  Dan Rosen, David Saunders,
  https://www.researchgate.net/publication/222695088_Risk_factor_contributions_in_portfolio_credit_risk_models
- “Capital Allocation to Business Units and Sub-Portfolios: the Euler Principle”(2008)
  Dirk Tasche, https://arxiv.org/abs/0708.2542
- “Relative importance of risk sources in insurance systems” (1998)
  North American Actuarial Journal, Volume 2, Issue 2
  Edward Frees, http://dx.doi.org/10.1080/10920277.1998.10595694

Contact details

- If you know of other ways to split or – even better – a new way to allocate, let me know!

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